Due July 29, 11:59pm on Gradescope.

The following are warm-up exercises and are *not* to be turned in. You may treat these as extra practice problems.

7.1.15, 7.1.28, 7.2.9, 7.2.23, 7.2.31, 7.3.5, 7.3.9, 7.4.29, 7.4.31, 7.4.35, 7.4.43, 9.5.16, 9.5.17.

Turn in the following exercises. Remember to carefully justify every statement that you write, and to follow the style of proper mathematical writing. You may cite any result proved in the textbook or lecture, unless otherwise mentioned. Each problem is worth 10 points with parts weighted equally, unless otherwise mentioned.

1. (**25 points**) Some examples involving explicit calculations:
   (a) (**5 points**) 7.2.26. Assume the probability distribution over all bit strings is uniform.
   (b) (**5 points**) 7.2.34(d).
   (c) (**5 points**) 7.3.10(d).
   (d) (**5 points**) 7.4.12(b). Please carry out the full computation with the infinite series. In particular, do not just cite Theorem 4 in Section 7.4 or use "series tables."
   (e) (**5 points**) 7.4.28.

2. Let X be a nonnegative random variable on a sample space S (i.e. $X(s) \geq 0$ for all $s \in S$). If $a > 0$, show that
$$P(X \geq a) \leq \frac{E(X)}{a}.$$
This is called *Markov's inequality.* Use Markov's inequality to deduce Chebyshev's inequality as a special case. [Hint for the first part: mimic the proof of Chebyshev's theorem and split the sum into two parts. Hint for the second part: given a random variable X, consider $(X - E(X))^2$.]

3. Use Chebyshev's inequality to prove the *weak law of large numbers*: suppose we have a sequence $X_1, X_2, \ldots$ of (mutually) independent random variables, such that $E(X_i) = \mu$ and $V(X_i) = \sigma^2$ for all i. Define $\overline{X}_n$ to be the random variable
$$\overline{X}_n = \frac{1}{n}(X_1 + X_2 + \ldots + X_n),$$
the average of the first n random variables. Prove that for any $\epsilon > 0$,
$$\lim_{n\to\infty} P(|\overline{X}_n - \mu| < \epsilon) = 1.$$

Remark: In other words, as the number of experiments goes to infinity, the probability that the average value of the experiment outputs is within ϵ of the mean approaches 1 (e.g. if you play a game with negative expected value at the casino, you are guaranteed to lose money in the long run).

4. (**15 points**) The *Riemann zeta function*, defined as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$, is perhaps the most important function in number theory. We will prove the *Euler product formula* for the zeta function using probability.

   (a) (**10 points**) Fix a real number $s > 1$. Show that the function $P(n) = n^{-s}/\zeta(s)$ is a probability distribution on the sample space $\mathbf{N}$. Let E_k be the event $\{n \in \mathbf{N} : k|n\}$. Show that if $p_1, p_2, \ldots$ are the prime numbers, then the infinitely many events $E_{p_1}, E_{p_2}, \ldots$ are *mutually* independent, in the sense that any finite subset of those events are mutually independent.

   (b) (**5 points**) Using the fact that 1 is the only positive integer not divisible by any prime, prove the Euler product formula:

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \zeta(s).$$

   Above, the left-hand side is an infinite product extending over all prime numbers. Feel free to extend results that we have discussed in class to the case of infinitely many events, and also feel free to ignore issues of convergence of infinite products/sums.

5. (**15 points**)

   (a) (**5 points**) Let $\sim$ be the relation on the set $\mathbf{R}$ such that $a \sim b$ if and only if $a - b \in \mathbf{Z}$. Show that $\sim$ is an equivalence relation, and describe the equivalence classes of $\sim$ (i.e. we are looking for an answer like "each equivalence class is of the form...").

   (b) (**10 points**) Give (with proof) an explicit bijection from $(\mathbf{R}/\sim)$ to the unit circle $S^1 = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$. Make sure to prove that your function is well-defined! [Hint: think about angles. You may have to scale by 2π at some point.]

6. (**15 points**)

   (a) (**5 points**) Let S and T be nonempty sets, and let $f : S \to T$ be a function. Let $\sim$ be the relation on S such that $s \sim s'$ if and only if $f(s) = f(s')$. Show that $\sim$ is an equivalence relation, and describe the equivalence classes of $\sim$.

   (b) (**10 points**) Give (with proof) an injection $(S/\sim) \to T$ that has image $\text{Im}(f)$ [Hint: you will want to define this function in terms of f]. Make sure to prove that your function is well-defined!

Remark: Part (b) is essentially the statement of the first isomorphism theorem from abstract algebra, rank-nullity from linear algebra, etc.

7. **(Bonus Problem, 10 points)** Let $E_1, E_2, \ldots$ be a countably infinite sequence of events in some probability space S. Let $\limsup_{n\to\infty} E_n$ be the subset of outcomes $s \in S$ that occur infinitely often in the E_i (i.e. it is the subset of $s \in S$ such that $s \in E_i$ for infinitely many E_i).

   (a) (**7 points**) Prove that if the sum $\sum_{i=1}^{\infty} P(E_i)$ of the probabilities of the E_i's is *finite* (i.e. the series converges), then $P(\limsup_{n\to\infty} E_n) = 0$; i.e. the probability that infinitely many of the events E_i occur is 0. This is called the (first) *Borel-Cantelli Lemma*. [Hint: show that $\limsup_{n\to\infty} E_n = \bigcap_{i=1}^{\infty} (\bigcup_{k=i}^{\infty} E_k)$.]

   (b) (**3 points**) Suppose Alan the gambler challenges you to a sequence of n games. In the ith game, you lose 2^i dollars with probability $1/(2^i + 1)$, and win 1 dollar with probability $2^i/(2^i + 1)$. For each i, compute the expected win/loss of the ith game. Then use the first Borel-Cantelli Lemma to determine if, as $n \to \infty$ (i.e. as you keep playing more and more games), the expected total loss is finite or infinite. So, should you keep playing with Alan "in the long run"?

   Remark: Note that the above is more or less equivalent to the "martingale" betting strategy. A word of warning: do ***not*** go to the casino and try this, because among other things, you have neither infinite time nor money (*Doob's optional stopping theorem* proves that such a strategy cannot "beat the house" in the long run).